NONLINEAR AND LINEAR WAVE PROCESSES IN ELECTROHYDRODYNAMICS TAKING ACCOUNT OF CHARGE DIFFUSION

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The investigation performed below is applicable to nonstationary electrohydrodynamic flows in which charged-particle diffusion processes play an essential part. Examples of such flows are nonstationary flows with moving electric charge fronts [1, 2], flows with gasdynamic parameter discontinuities, boundary layers near electrode grids, flows with electrification of bodies during motion in streams, the phenomenon of electric charge formation and methods to control it in different technological processes [2], etc. The extraordinary electrohydrodynamic situation also occurs during combustion in an electric field [3], for instance, when a laminar flame front is placed between plane electrodes [4, 5]. In this case, charged domains of opposite charge are formed on either side of the combustion front (between the electrode and the flame). The massive electrical forces acting on the flow hydrodynamics during combustion result in different electrohydrodynamic effects into whose description the charge diffusion must be taken into account especially near the combustion front. By varying the applied voltage, different nonstationary transients can be obtained with the adjustment of the electrical and hydrodynamical parameters in the charged domains and can act effectively on the combustion front.

1. BURGERS EQUATION FOR THE ELECTRIC FIELD INTENSITY

The transfer and dissipation processes due to viscosity and heat conductivity are considered inessential in the examples presented above; hence, we shall utilize the following system of equations [1, 2, 4] to describe electrohydrodynamic flows with charged-particle diffusion taken into account:

$$\partial \mathbf{E}/\partial t + 4\pi \mathbf{j} = (\mathrm{rot}\ \mathbf{H})c; \tag{1.1}$$

$$\mathrm{rot}\ \mathbf{E} = 0; \tag{1.2}$$

$$\mathrm{div}\ \mathbf{E} = 4\pi q; \tag{1.3}$$

$$\mathbf{j} = q(b\mathbf{E} + \mathbf{v}) - D\,\mathrm{grad}\, q; \tag{1.4}$$

$$\partial \rho/\partial t + \mathrm{div}\ \rho\mathbf{v} = 0; \tag{1.5}$$

$$\rho d\mathbf{v}/dt + \mathrm{grad} p = q\mathbf{E}; \tag{1.6}$$

$$p = \rho RT; \tag{1.7}$$

$$C_V \rho dT/dt + p\ \mathrm{div}\ \mathbf{v} = (\mathbf{j} - q\mathbf{v})\mathbf{E}, \tag{1.8}$$

where **v**, **E**, **H**, **j** are the vectors of the velocity of the medium, the electric and magnetic field intensities, and the current density ρ, p, T are the density, pressure and temperature, q is the volume electric charge density, R is the gas constant, C_V is the specific heat at constant volume, D and b are diffusion coefficients, and c is the speed of light.

Applying the divergence operation to both sides of (1.1) we obtain the law of charge conservation

$$\partial q/\partial t + \mathrm{div}\ \mathbf{j} = 0,$$

which is, together with (1.2)-(1.8), the known system of electrohydrodynamics equations [1, 2]. Eliminating j and q from (1.1)-(1.4), we obtain an equation for the electric field intensity

$$\partial \mathbf{E}/\partial t + (\mathbf{v} + b\mathbf{E})\mathrm{div}\ \mathbf{E} - D\Delta\mathbf{E} = \mathrm{rot}\ \mathbf{H}c. \tag{1.9}$$

Assuming the flow to possess some spatial symmetry, we set the right side of the equation identically equal to zero. In the case of a weak electrohydrodynamic (EHD) interaction (the right sides of (1.6) and (1.8) are zero), we can consider that v = const. Then (1.9) is the three-dimensional analog of the Burgers equation, known in hydrodynamics [6-8], which describes the origination and formation of a shock in which charged-particle diffusion plays the part of viscosity. Going over to an inertial coordinate system, we obtain in the one-dimensional case

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$$\partial E/\partial t + bE\partial E/\partial t = D\partial^2 E/\partial x^2. \tag{1.10}$$

which by using the Hopf substitution [9]

$$E = -\frac{2D}{b}\frac{\partial}{\partial x}\ln F$$

reduces to the linear equation of heat conduction

$$\partial E/\partial t = D\partial^2 F/\partial x^2.$$

Then the solution of (1.10) with the initial condition $E(x, 0) = E_0(x)$ can be written in the form

$$E = -\frac{2D}{b}\frac{\partial}{\partial x}\ln\left\{\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{+\infty}\exp\left[-\frac{(x-\xi)^2}{4Dt} - \frac{b}{2D}\int_0^{\xi}E_0(\eta)\,d\eta\right]d\xi\right\} = \frac{1}{b}\frac{x}{t} - \frac{1}{bt}\times$$
$$\times\frac{\int_{-\infty}^{+\infty}\xi\exp\left\{-\frac{(x-\xi)^2}{4Dt} - \frac{b}{2D}\int_0^{\xi}E_0(\eta)\,d\eta\right\}d\xi}{\int_{-\infty}^{+\infty}\exp\left\{\frac{(x-\xi)^2}{4Dt} - \frac{b}{2D}\int_0^{\xi}E_0(\eta)\,d\eta\right\}d\xi}. \tag{1.11}$$

The necessary condition for convergence of the integral in (1.11) is

$$b\int_0^x E_0(\eta)\,d\eta \leqslant \text{const}\,x, \quad x\to\infty.$$

Considering the evolution of the initial perturbations, which damp as $x \to \pm\infty$, we put

$$b\int_{-\infty}^{+\infty}E_0(\eta)\,d\eta \equiv b\Phi < \infty.$$

For any t

$$\int_{-\infty}^{+\infty}E(x, t)\,dx = \int_{-\infty}^{+\infty}E_0(\eta)\,d\eta \equiv \Phi. \tag{1.12}$$

Let us note that for the invariant (1.12), which is the integral of motion for the Burgers equation (1.10), there is a potential difference in the initial perturbations at the ends of the time-conserving segment of integration in our case. Taking account of

$$\int_{-\infty}^{0}E_0(x)\,dx = \int_0^{+\infty}E_0(x)\,dx = \frac{1}{2}\int_{-\infty}^{+\infty}E_0(x)\,dx = \frac{\Phi}{2}$$

the asymptotic form of the solution (1.11) as $t\to\infty$

$$\underset{t\to\infty}{E}(x, t) = -2\frac{D}{b}\frac{d}{dx}\ln\left[e^{-\frac{b\Phi}{4D}}\int_{-\infty}^{\frac{x}{\sqrt{4Dt}}}e^{-\eta^2}\,d\eta + e^{+\frac{b\Phi}{4D}}\int_{\frac{x}{\sqrt{4Dt}}}^{+\infty}e^{-\eta^2}d\eta\right]. \tag{1.13}$$

It follows from (1.13) that as $t\to\infty$ the solution depends only on the magnitude of the electric potential of the initial perturbation.

As $D\to 0$ the asymptotic formula (1.13) is simplified

$$\underset{\substack{t\to\infty\\ D\to 0}}{E}(x, t) = \frac{1}{b}\frac{x}{t}\sqrt{Dt}\,\frac{e^{\frac{b\Phi}{2D}}+1}{\sqrt{Dt}\left(e^{\frac{b\Phi}{2D}}-1\right)+\sqrt{\pi}\,x\,e^{\frac{x^2}{4Dt}}},$$

which yields in the limit as $D\to 0$

$$\lim_{D\to 0}E(x, t\to\infty) = \begin{cases}\frac{1}{b}\frac{x}{t}, & 0 < x < \sqrt{2b\Phi t},\\ 0, \quad x < 0, & x > \sqrt{2b\Phi t}.\end{cases}$$

2. STATIONARY SOLUTION OF THE BURGERS EQUATION

The Burgers equation also allows a stationary solution in the form of a traveling wave being propagated without deformation at a constant velocity W. Substituting $E = f(x - Wt) = f(\xi)$ and being interested in bounded solutions, we obtain for the wave being propagated to the right

$$E(\xi) = \frac{W}{b} + c_1 - \frac{2c_1}{1 - c_2 \exp\left[-\frac{bc_1\xi}{D}\right]} \quad (2.1)$$

(c_1 and c_2 are constants of integration). The solution (2.1) is a certain wave of the electrical field intensity which is similar to a shock or rarefaction wave. The magnitude of the jump and the characteristic value of the transition domain equal $2c_1$ and D/bc_1, respectively, i.e., are determined from the boundary conditions of the problem. In order to illustrate the properties of a stationary wave, let us be given (although formally) the following conditions:

$$x = 0,\ E = E_1,\ x \to \infty,\ E = E_0. \quad (2.2)$$

We hence obtain

$$c_1 = \frac{W}{b} - E_0, \quad c_2 = \frac{E_1 - E_2}{E_1 - 2\frac{W}{b} + E_0}.$$

The velocity of stationary wave propagation W hence remains arbitrary. In order to determine it, certain additional considerations are needed. Either the electrical charge density being propagated together with the wave (i.e., the derivative of the electrical field with respect to the coordinate), or the rate of its formation in an infinitely narrow zone of ion generation in the wave front must be given as an additional condition to (2.2). By setting

$$x = 0, \quad q_0 = \frac{\varepsilon_0}{4\pi} \frac{dE}{dx}\bigg|_{x=0} = -\frac{b\varepsilon_0}{2\pi D} \frac{c_1^2 c_2}{(1 - c_2)^2},$$

we obtain

$$W = \frac{b(E_1 + E_0)}{2} + \frac{4\pi D}{\varepsilon_0} \frac{q_0}{E_1 - E_0}, \quad (2.3)$$

$$c_1 = \frac{E_1 - E_2}{2} + \frac{4\pi D}{b\varepsilon_0} \frac{q_0}{E_1 - E_0}, \quad c_2 = \frac{b\varepsilon_0}{8\pi D} \frac{(E_1 - E_0)}{q_0}.$$

It is seen from (2.3) that the half-width $2c_1$ and the velocity of wave propagation W depend on the values E_1 and E_0 and the properties of the charged gas.

Let us make numerical estimates. For example, the characteristic values of the electrical parameters on each side of the combustion front are the following during combustion in an electrical field: $E_1 \sim 10^3$ V/cm, $E_0 \sim 10$ V/cm, $q_0 \sim 10^9$ cm$^{-3} \cdot$ e (e is the charge on an electron), $b \sim 1$ cm^2/V$\cdot$sec, $D \sim 1$ cm^2/sec, $\varepsilon_0 \sim 1$. Then the magnitude of the jump is $2c_1 \sim 10^3$ V/cm, the half-width of the wave is $\Delta = D/bc_1 \sim 2 \cdot 10^{-3}$ cm, and the wave velocity is $W \sim 5 \cdot 10^2$ cm/sec. It is assumed in these estimates that the space charge is formed by ions. When the majority of charge carriers are electrons, by setting $E_1 \sim 10^2$ V/cm, $E_0 \sim 0$, $q_0 \sim 10^9$ cm$^{-3} \cdot$ e, $b \sim 10^3$ cm^2/V$\cdot$sec, $D \sim 10^3$ cm^2/sec, $\varepsilon_0 \sim 1$, we obtain $2c_1 \sim 1.4 \cdot 10^2$ V/cm, $\Delta = D/bc_1 \sim 1.4 \cdot 10^{-2}$ cm, $W \sim 7 \cdot 10^4$ cm/sec. As is seen from the estimates presented, the magnitude of the jump and the half-width of the wave are determined mainly by the electrical field intensity in front of and behind the wave front, while the wave propagation velocity depends essentially on the mobility of the charged particles being formed in the wave because of ionization, or formed in the initial instant by some source.

Let us note that the investigation presented above can be applicable to nonstationary electrohydrodynamic flows when the velocity of the medium v in (1.9) is determined from the flow hydrodynamics, i.e., in the approximation of weak electrohydrodynamic interaction.

3. LINEAR EHD WAVES TAKING ACCOUNT OF CHARGE DIFFUSION

Now, let us examine the case when it is impossible to neglect electrohydrodynamic interaction.

In the case when the electrical mass forces exert a substantial effect on the flow hydrodynamics, the velocity of the medium depends on the magnitude of the field intensity and is determined from (1.5)-(1.8). In the one-dimensional case, we obtain for curl $H \equiv 0$

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\rho v = 0, \quad \rho\frac{d}{dt}v = -\frac{\partial}{\partial x}\left(p - \frac{E^2}{8\pi}\right), \quad p = \rho RT,$$

$$C_V \rho \frac{d}{dt} T + p\frac{\partial}{\partial x} v = \frac{b}{4\pi} E^2 \frac{\partial}{\partial x} E - \frac{D}{4\pi} E \frac{\partial^2}{\partial x^2} E, \quad (3.1)$$

$$\frac{\partial E}{\partial t} + (v + bE)\frac{\partial E}{\partial x} = \frac{\partial^2 E}{\partial x^2}.$$

The system (3.1) is a complete system of electrohydrodynamic equations for one-dimensional flows with charge diffusion taken into account. Linear electrohydrodynamic waves without diffusion were considered in [5, 10]. In particular, acoustic wave propagation in charged media being formed during combustion in an electrical field was investigated in [5].

Let us linearize the system (3.1) by denoting the parameters of the unperturbed state by a zero subscript and considering them constants, while the perturbations will be denoted by primed quantities. In a coordinate system moving at the velocity v_0 we obtain

$$\frac{\partial E'}{\partial t} + bE_0\frac{\partial E'}{\partial x} = \frac{\partial^2 E'}{\partial x^2}; \tag{3.2}$$

$$\frac{\partial^2 p'}{\partial t^2} = a^2\frac{\partial^2 p'}{\partial x^2} - \frac{E_0}{4\pi}\left[a^2\frac{\partial^2 E'}{\partial x^2} - (\gamma-1)\, bE_0\frac{\partial^2 E'}{\partial t\partial x} + (\gamma-1)\, D\frac{\partial^3 E'}{\partial t\partial x^2}\right], \tag{3.3}$$

where $a=\sqrt{\gamma p_0/\rho_0}$ is the thermodynamic speed of sound, and γ is the adiabatic index.

The usual acoustics equation follows from (3.3) in the absence of an electrical field ($E_0=0$).

As is seen from (3.2), in a linear approximation with $v=\text{const}$ the gasdynamic parameters exert no influence on the field formation. The solution of (3.2), which is a linear heat conduction equation, can be written in the form

$$E'(x, t) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{+\infty} e^{-\frac{1}{4Dt}(\xi - x + bE_0 t)^2} E'(\xi, 0)\, d\xi, \tag{3.4}$$

where $E'(\xi, 0) = E'(\xi, \tau)|_{\tau=0}$ is the initial perturbation.

Now (3.3) is an inhomogeneous wave equation in which $E'(x, t)$ is expressed from (3.4). Let us write the solution of this wave equation

$$p'(x, t) = \frac{\varphi(x+at)+\varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\, d\xi + \frac{1}{2a}\int_0^t d\tau\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi, \tau)\, d\xi, \tag{3.5}$$

$$\varphi(\xi) = p'(\xi, 0), \quad \psi(\xi) = \frac{\partial}{\partial\tau}p'(\xi, \tau),$$

$$f(\xi, \tau) = -\frac{E_0}{4\pi}\left[a^2\frac{\partial^2 E'}{\partial\xi^2} - (\gamma-1)\, bE_0\frac{\partial^2 E'}{\partial\tau\partial\xi} + (\gamma-1)\, D\frac{\partial^3 E'}{\partial\tau\partial\xi}\right].$$

In the particular case when the medium is described by a polytropic law of state

$$p = \text{const}\ \rho^n, \tag{3.6}$$

where n is the polytropy index, the system of equation simplifies substantially: Instead of the third and fourth equations from (3.1) we write the relationship (3.6). In this case we also obtain a system of equations analogous to (3.2) and (3.3) with the sole difference that the function $f(\xi, \tau)$ in the right side of the inhomogeneous wave equations (3.3) and its solution (3.5) has the form

$$f(\xi, \tau) = -\frac{np_0}{\rho_0}\frac{E_0}{4\pi}\frac{\partial^2 E'}{\partial\xi^2}.$$

If we seek the solution of (3.2) and (3.3) in the form of plane waves ($p'(x, t) = p'_* e^{i(kx-\omega t)}$, $E'(x, t) = E'_* e^{i(kx-\omega t)}$ [11, 12], where p'_* and E'_* are constants), then a dispersion relationship

$$(\omega - bE_0 k + iDk^2)(\omega^2 - a^2k^2) = 0. \tag{3.7}$$

can be obtained by using standard operations.

This dispersion relationship determines three waves: Two acoustic waves being propagated to opposite sides with the velocity a, and the electrical field intensity wave whose phase velocity equals

$$\frac{\text{Re}\,\omega}{\text{Re}\,k} = bE_0 + 2D\,\text{Im}\,k.$$

Let us note that the linearized system (3.2), (3.3) is valid, strictly speaking, just for a quasineutral medium in the unperturbed state ($E_0=\text{const}$).

If the medium has the space charge $q_0 \neq 0$, the stationary electrical field intensity, pressure, density, and temperature distributions in the unperturbed state are then subject to the following relations (for $v_0=0$):

$$E_0 = c_0 - \frac{2c_0}{1-\exp\left(-\frac{bc_0x}{D}+c_1\right)}, \quad p_0 = c_2 + \frac{E_0^2}{8\pi},$$
$$\rho_0 = \frac{c_2 + \frac{E_0^2}{8\pi}}{RT_0}, \quad T_0 = c_3, \tag{3.8}$$

where c_0, c_1, c_2, c_3 are constants of integration. These relationships follow from the system (3.1) written in the stationary case for $v_0 = 0$.

If the unperturbed parameters of the medium are determined from (3.8), then the linearized system (3.1) has the form

$$\frac{\partial E'}{\partial t} + bE'\frac{\partial E_0}{\partial x} + bE_0\frac{\partial E'}{\partial x} - D\frac{\partial^2 E'}{\partial x^2} + v'\frac{\partial E_0}{\partial x} = 0,$$
$$\frac{\partial \rho}{\partial t} + \rho_0\frac{\partial v'}{\partial x} = 0, \quad \rho_0\frac{\partial v'}{\partial t} + \frac{\partial p'}{\partial x} - \frac{1}{4\pi}\left(E_0\frac{\partial E'}{\partial x} + E'\frac{\partial E_0}{\partial x}\right) = 0,$$
$$\frac{C_V\partial p'}{R\partial t} - C_VT_0\frac{\partial \rho'}{\partial t} + p_0\frac{\partial v'}{\partial x} - \frac{b}{4\pi}E_0^2\frac{\partial E'}{\partial x} - \frac{b}{2\pi}E_0E'\frac{\partial E_0}{\partial x} +$$
$$+\frac{D}{4\pi}E_0\frac{\partial^2 E'}{\partial x^2} + \frac{D}{4\pi}E'\frac{\partial^2 E_0}{\partial x^2} = 0.$$

Seeking the solution of this system in the form of plane waves, and equating the determinant of this system to zero, we can obtain the following dispersion relation:

$$\omega^3 + \left[iDk^2 - bE_0k + ib\frac{\partial E_0}{\partial x}\right]\omega^2 - \left[a^2k^2 + \frac{i}{4\pi}\frac{E_0}{\rho_0}\frac{\partial E_0}{\partial x}k + \frac{1}{4\pi\rho_0}\left(\frac{\partial E_0}{\partial x}\right)^2\right]\omega -$$
$$- iDa^2k^4 + E_0\left[a^2b - \frac{D}{4\pi}\frac{\gamma-1}{\rho_0}\frac{\partial E_0}{\partial x}\right]k^3 - ib\left[a^2 + \frac{\gamma-1}{4\pi}\frac{E_0^2}{\rho_0}\right]k^2 + \frac{\gamma-1}{4\pi\rho_0}\frac{\partial E_0}{\partial x}\left[D\frac{\partial^2 E}{\partial x^2} - 2bE_0\frac{\partial E_0}{\partial x}\right]k = 0. \tag{3.9}$$

One of the four solutions of the dispersion relationship is the trivial solution $\omega = 0$, which is not included in (3.9). It is also evident that the dispersion relation (3.7) is a particular case of (3.9) with E_0=const. Without writing down the explicitly awkward solutions of the cubic equation, let us just present some approximate results.

In particular, we have for the short waves of the high-frequency branch

$$\omega^3 = iDa^2k^4, \tag{3.10}$$

and for the long waves

$$\omega^3 = \frac{1-\gamma}{2\pi\rho_0}\left[\frac{D}{4}\frac{\partial}{\partial x}\left(\frac{\partial E_0}{\partial x}\right)^2 - bE_0\left(\frac{\partial E_0}{\partial x}\right)^2\right]k.$$

The dispersion relationship for the short waves in the low-frequency branch has the form

$$\omega = -iDk^2, \tag{3.11}$$

while for the long waves

$$\omega = (\gamma - 1)\left[D\left(\frac{\partial E_0}{\partial x}\right)^{-1}\frac{\partial^2 E_0}{\partial x^2} - 2bE_0\right]k.$$

Analyzing the relationships written down, the deduction can be made that charge diffusion processes play the governing role for short waves. This result is perfectly natural since the terms with diffusion are dominant for fine-scale phenomena.

In the case when the medium is subjected to the polytropic law of state (3.6), the dispersion relation becomes

$$\omega^3 + i\left(Dk^2 + ibE_0k + b\frac{\partial E_0}{\partial x}\right)\omega^2 - \left[n\frac{p_0}{\rho_0}k^2 + i\left(\frac{E_0}{4\pi\rho_0}\frac{\partial E_0}{\partial x} - n\frac{\partial}{\partial x}\frac{p_0}{\rho_0}\right)k +\right.$$
$$\left.+\frac{1}{4\pi\rho_0}\left(\frac{\partial E_0}{\partial x}\right)^2\right]\omega - \left(Dk^3 + ibE_0k^2 + b\frac{\partial E_0}{\partial x}k\right)\left(in\frac{p_0}{\rho_0}k + n\frac{\partial}{\partial x}\frac{p_0}{\rho_0}\right) = 0.$$

Separating the high- and low-frequency branches of the solution, we obtain for the short waves in a high-frequency branch

$$\omega^3 = iDn\frac{p_0}{\rho_0}k^4, \tag{3.12}$$

while for the long waves

$$\omega^3 = bn \frac{\partial E_0}{\partial x} \frac{\partial}{\partial x} \left(\frac{p_0}{\rho_0} \right) k.$$

Correspondingly, the dispersion equation for the short waves in the low-frequency branch has the form

$$\omega = -iDk^2, \tag{3.13}$$

while for long waves

$$\omega = 4\pi\rho_0 bn \left(\frac{\partial E_0}{\partial x} \right)^{-1} \frac{\partial}{\partial x} \left(\frac{p_0}{\rho_0} \right) k.$$

Let us note the agreement between the results for the short waves in the case when the medium is subjected to a polytropic law of state, and in the general case when the energy equation is written down. In particular, there will be complete correspondence between (3.10) and (3.12) if the polytropic speed of sound is introduced as $a = \sqrt{np_0/\rho_0}$ and (3.11) agrees with (3.13). It should be noted that the polytropic law describes the state of the medium sufficiently well, for instance, for adiabatic processes proceeding so rapidly that heat does not succeed in being transmitted from one point to another.

In conclusion, let us note that the analysis performed and the results obtained are valid for unipolarly charged flows in which interaction processes between a charged medium and an electric field as well as charge diffusion play a substantial part.

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